Analysis of acoustic transmission for one directional periodic bounded structure in 2D by BEM

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In this paper, the acoustic transmission of one directional 2D finite periodic bounded structure is investigated by BEM, and to avoid solving the whole matrix, a size-reduced system matrix is formulated to compute the transmission problem. In the size-reduced system matrix, the unknowns on free boundary of the periodic parts are removed. Also the block Sakurai-Sugiura (SS) method is employed to compute the band structure of a unit cell for the infinite periodic structure. In order to eliminate the fictitious eigenvalues resulted by internal closed boundary, Burton-Miller’s method is used. Furthermore, we compute the eigenfrequencies of the finite structure having periodic parts and show that the distribution of the eigenfrequencies follows the location of bandgap. Numerical results show that when the number of layers of the periodic parts in the finite periodic structure is large enough, the transmissivity is in a good agreement with the bandgaps.

Key Words: finite periodic structure, block Sakurai-Sugiura Method, eigenfrequency, boundary element method

1. Introduction

The studies of 1D periodic structures or materials have received considerable attention in the literature1, 2, 3, 4. Due to the dispersive nature of periodic structures, they may exhibit frequency range where the propagation of mechanical waves are forbidden. These frequency ranges are commonly called stop bands or bandgaps and the remaining ranges are called pass bands. To compute the dispersion curves and find the bandgaps, transfer matrix method2, 3, 4 is often used, however boundary element method (BEM) is also an attractive method to find bandgaps since it can keep the unknowns on the boundary where homogenous condition is given.

Block SS method5, 6, 7, 8, 9, 10, which is one of contour integral methods, converts the nonlinear eigenvalue problem (NEP) to a generalized eigenvalue problem of two Hankel matrices with a reduced eigenspace. The dimension of the Hankel matrices is very small and computational cost is neglectable comparing with the original system matrix. The main computational cost is, however, the evaluation of the moment matrices which are used to form the Hankel matrices. It results in solving BEM for N times which is the number of points on the contour integral path when trapezoidal rule is employed. The emerging of fast algorithm such as fast multipole boundary element method (FMBEM)11, adaptive cross-approximation boundary element method (ACA-BEM)12, 13 makes it possible to solve eigenvalue problem by using BEM efficiently.

We presented eigenvalue analysis for 2D cavities, in author’s previous research14, in which only simply connected region is considered, so there is no internal closed boundary or fictitious eigenvalues exist in the numerical results.

In this research, we apply the block SS method to compute the eigenvalues of the transmission problems in one directional periodic bounded structure in two dimensional domain. To reduce the dimension of the system matrix, we only carry out the integration of unit cell once and represent the unknowns on the free boundary by using the unknowns on connection interfaces which are considered as virtual interfaces if the same materials are used at the connection interfaces of adjacent cells, but if the materials are not same, then the interfaces are not virtual. For the calculation of dispersion curves, the scatter in the matrix material has a closed boundary which means that the domain is a complex connected region. The internal closed boundary results in appearance of fictitious eigenvalues when the eigenvalue problem is solved by using block SS method with conventional BEM. In order to acquire the correct results without any fictitious eigenvalues, we here employ Burton-Miller’s method15, by which large imaginary components are added to the fictitious eigenvalues, then they can be filtered because of the large imaginary part. Numerical examples with the different number of finite periodic layers are given to demonstrate the correctness of proposed methodology.

2. Formulation

In physics, the propagation of acoustic waves through a material medium is governed by the Helmholtz equation:

\[ \nabla^2 p(x) + k^2 p(x) = 0 \quad \text{in } T, \]  

(1)
where \( p \) is the sound pressure, \( \nabla^2 \) is the Laplace operator, and \( k = \frac{\omega}{C} \) is a predefined wave number, \( \omega \) is circular frequency, \( C \) is the wave speed, and \( T \) is the domain. Usually we have the following boundary conditions:

\[
p = \bar{p} \quad \text{on} \quad S_p,
q = \frac{\partial \bar{p}}{\partial n} = i \rho \omega \bar{v} \quad \text{on} \quad S_q,
\]

where \( n \) is the normal direction at the boundary, \( \rho \) is the density, \( v \) is the particle velocity.

Equation (1) can be transformed into the integral equation by using the following fundamental solution (2) and its normal derivative (3) for 2D Helmholtz equation:

\[
p^*(x,y) = \frac{1}{4} H_0^{(1)}(kr), \quad \text{(2)}
\]

\[
q^*(x,y) = \frac{\partial p^*(x,y)}{\partial n(x)} = -\frac{k_i}{4} H_1^{(1)}(kr) \frac{\partial r}{\partial n(x)}, \quad \text{(3)}
\]

where \( y \) is the collocation point and \( x \) is the source point, \( r \) is the distance between \( x \) and \( y \), \( i \) is the imaginary unit, \( H_1^{(1)} \) denotes the Hankel function of the first kind, \( n(x) \) is the outward normal direction to the boundary at \( x \). With Green’s identity, the integral representation for \( p(y) \) can be obtained as

\[
p(y) + \int_S q^*(x,y) p(x) dS(x) = \int_S p^*(x,y) q(x) dS(x). \quad \text{(4)}
\]

The normal directional derivative of \( p(y) \) at \( y \) is written as

\[
q(y) + \int_S \hat{q}^*(x,y) p(x) dS(x) - \int_S \hat{p}^*(x,y) q(x) dS(x) = 0 \quad \text{in} \ T, \quad \text{(5)}
\]

where \( \hat{p}^*(x,y) \) and \( \hat{q}^*(x,y) \) are the normal directional derivative of \( p^*(x,y) \) and \( q^*(x,y) \), respectively:

\[
\hat{p}^*(x,y) = \frac{\partial p^*(x,y)}{\partial n(x)} = -\frac{k_i}{4} H_1^{(1)}(kr) \frac{\partial r}{\partial n(x)},
\]

\[
\hat{q}^*(x,y) = \frac{\partial q^*(x,y)}{\partial n(x)} = -\frac{k_i}{4} H_1^{(1)}(kr) \frac{\partial q}{\partial n(x)}
\]

\[
-\frac{k_i}{4} H_1^{(1)}(kr) \frac{\partial q}{\partial n(x)} - \frac{k_i}{4} H_1^{(1)}(kr) \frac{\partial q^*}{\partial n(x)}, \quad \text{(6)}
\]

Let the internal point \( y \) tend to the boundary of \( T \), then, evaluating the boundary integrals in Eqs. (4) and (5), we obtain the following boundary integral equations:

\[
c_v p(y) + \int_S q^*(x,y) p(x) dS(x) = \int_S p^*(x,y) q(x) dS(x), \quad \text{(8)}
\]

\[
c_v q(y) + \int_S \hat{q}^*(x,y) p(x) dS(x) = \int_S \hat{p}^*(x,y) q(x) dS(x), \quad \text{(9)}
\]

where \( c_v = \frac{1}{2} \) if \( S \) is smooth around \( y \), \( f \) denotes that the integral must be evaluated in the sense of Cauchy’s principal value, \( f \) denotes that the integral must be evaluated in the sense of finite-part value of divergent integral.

Let us consider two domains connected by a certain number of periodic structures shown in Fig.1, in which \( \Omega_i \) denotes the input domain, \( \Omega_o \) denotes the output domain, \( D_1, D_2, \ldots, D_o \) are the matrix domains of the periodic parts, \( \Sigma_1, \Sigma_2, \ldots, \Sigma_o \) are scatters in the corresponding matrix domains.

First, let us derive the system equations of a unit cell in the periodic parts. In Fig.2, the boundary of the \( l \)-th cell is defined as follows: the boundary shown with a broken line on the left-hand side is the output boundary \( \Gamma_{l-1}^o \) of cell \( l-1 \) and also the input boundary \( \Gamma_1^i \) of cell \( l \); similarly, the boundary shown with a broken line on the right-hand side is the output boundary \( \Gamma_l^o \) of cell \( l \) and also the input boundary \( \Gamma_{l+1}^i \) of cell \( l+1 \). \( \Gamma_l^i \) denotes the remaining free part of the boundary.

The discretized boundary integral equations of a unit cell can be written in a matrices form as follows:

\[
\begin{pmatrix}
H_{ii} & H_{io} & H_{is} \\
H_{oi} & H_{oo} & H_{os} \\
H_{si} & H_{so} & H_{ss}
\end{pmatrix}
\begin{pmatrix}
p_i \\
p_o \\
p_s
\end{pmatrix}
=
\begin{pmatrix}
G_{ii} & G_{io} & G_{is} \\
G_{oi} & G_{oo} & G_{os} \\
G_{si} & G_{so} & G_{ss}
\end{pmatrix}
\begin{pmatrix}
q_i \\
q_o \\
q_s
\end{pmatrix}. \quad \text{(10)}
\]

In Eq.(10), \( p_i, p_o, p_s \), and \( q_i, q_o, q_s \) are the sound pressures and its normal derivatives on the input boundary \( \Gamma_i \), the output boundary \( \Gamma_o \), and the free boundary \( \Gamma_s \), respectively.

Rearranging the coefficients of the quantities on the free boundary yields

\[
\begin{pmatrix}
H_{ii} & H_{io} & A_{is} \\
H_{oi} & H_{oo} & A_{os} \\
H_{si} & H_{so} & A_{ss}
\end{pmatrix}
\begin{pmatrix}
p_i \\
p_o \\
p_s
\end{pmatrix}
=
\begin{pmatrix}
G_{ii} & G_{io} & B_{is} \\
G_{oi} & G_{oo} & B_{os} \\
G_{si} & G_{so} & B_{ss}
\end{pmatrix}
\begin{pmatrix}
q_i \\
q_o \\
q_s
\end{pmatrix}, \quad \text{(11)}
\]

where \( x_s \) denotes the unknowns, \( y_s \) denotes the knowns on the free boundary. Then, \( x_s \) can be represented by other quantities, as follows:

\[
x_s = -A_{is}^{-1} H_{is} p_i - A_{is}^{-1} H_{io} p_o \\
+ A_{is}^{-1} G_{oi} q_i + A_{is}^{-1} G_{so} q_o + A_{is}^{-1} B_{so} y_s. \quad \text{(12)}
\]
In light of (11) and (12), the unknowns on the free boundary are eliminated, as follows:

\[
\begin{pmatrix}
H_{io} - A_{io}A_{ss}^{-1}H_{so} & -G_{io} + A_{io}A_{ss}^{-1}G_{so} \\
H_{so} - A_{so}A_{ss}^{-1}H_{io} & -G_{so} + A_{so}A_{ss}^{-1}G_{io}
\end{pmatrix}
\begin{pmatrix}
p_o \\
q_o
\end{pmatrix}
\]

where \(H_{i/o}, G_{i/o}, A_{i/o} = 100\) layers of periodic cell. All unknowns on the free boundary of the connection parts are removed, for instance, if we consider 100 layers of periodic cell and the number of elements for free boundary of unit cell is 20, then 20 \(\times\) 100 = 2000 unknowns and equations are reduced.

We now consider an infinite one directional 2D periodic structure formed with the cells shown in Fig. 2. Then, the system of the algebraic equations can be written in the form:

\[
\begin{pmatrix}
H_i & H_o \\
L_i & L_o
\end{pmatrix}
\begin{pmatrix}
p_i \\
q_i
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad (18)
\]

According to Bloch’s theorem, the following relations hold:

\begin{align*}
p_o &= p_i \exp(ika), \\
q_o &= q_i(-\exp(ika)). \quad (19, 20)
\end{align*}

where \(a\) is the length of a single cell along the periodic direction, \(k\) is the wave vector. When we substitute above relations and the homogeneous boundary condition on the free boundary to Eq.(18), we have the following system of NEP:

\[
\begin{pmatrix}
H_i + H_o \exp(ika) & -G_i - G_o(-\exp(ika)) \\
G_i & G_o
\end{pmatrix}
\begin{pmatrix}
p_i \\
q_i
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad (21)
\]

As \(k = \frac{\omega}{v}\), the eigenvalue parameter \(\omega\) is involved in coefficient matrix of Eq. (21) implicitly. Here, we employ a contour integral method, the block SS method as an eigensolver which has been applied to solve the eigenvalues of a 2D acoustic cavity by BEM (14), and Burton-Miller’s method is used to eliminate the fictitious eigenvalues. The parameter \(\alpha\) of Burton-Miller’s method is chosen as \(i/2k\).

3. Numerical example

A finite structure having periodic parts is depicted in Fig.4. Square input and output domains are connected by \(N\) layers of square cells with cylinder scatters, and all sides of the square structures are assumed to be 1[m]. A nonsymmetrical excitation is given as the particle velocity \(v = 1[m/s]\) on 70% of the edge of the input domain, while in the output domain, 30 observation points are distributed. The points from number 1 to 10, 11 to 20 and 21 to 30 have the same \(y\) coordinates as \(y = 0.5, y = 0.25, y = 0.75\) respectively, while \(x\) coordinates for the each 10 points are 0.05, 0.15, \(\cdots\), 0.95, and unit is [m]. First, assuming that the unit cell is one layer of an infinite one directional periodic structure and solving the NEP given
Table 1 Parameters of the model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sound velocity [m/s]</td>
<td>337.2</td>
</tr>
<tr>
<td>Density [kg/m$^3$]</td>
<td>1.22</td>
</tr>
<tr>
<td>Filling fraction</td>
<td>0.503</td>
</tr>
<tr>
<td>Filling material</td>
<td>rigid</td>
</tr>
<tr>
<td>Cell length [m]</td>
<td>1</td>
</tr>
</tbody>
</table>

by Eq.(21) using BEM based on Burton-Miller’s method, we obtain the dispersion curves corresponding to the variable wave vector $k$ from 0 to $\pi$. For the Block SS method, solved range is assumed as [500,4500]. The numerical results for the dispersion curves obtained by the conventional BEM and that based on Burton-Miller’s method are shown in Figs.5 and 6, respectively. Since the fictitious eigenvalues are constants when the wave vector $k$ varies from 0 to $\pi$, there are horizontal lines showing fictitious eigenvalues in Fig.5.

Fig. 5 Dispersion curves (by conventional BEM) with fictitious eigenvalues (horizontal lines).

Next, let us consider the transmission problem and use different models by changing the layers of the connection parts. The transmissivities of the finite periodic structure with 12, 25, 50, 100 layers are shown in Figs.7 to 10, in which the sound pressures at 30 observation points are plotted, and in Figs.7(b) to 10(b) are shown their projections on $\omega$-$p$ plane. From the figures of transmissivity, it is found that the transmissivity becomes very low in the bandgaps particularly when the number of the layers is larger than 25.

To obtain the bandgaps of the finite structure, the block SS method is directly applied to compute the eigenfrequencies of the size-reduced system matrix (17) that is also of a NEP. The eigenfrequencies of models with 12, 25, 50, 100 layers are shown in Figs.11 to 14, respectively, corresponding to the solved range [500,1500] in which 1 bandgap is included. Eigenfrequencies out of this range are also obtained. From these numerical results, it is found that the distribution of the eigenvalues along the real-axis have the bandgap shown in Fig. 6. However, as is found in Figs.11 to 14, two eigenfrequencies 1252.513 and 1262.504 are always found in the range of bandgap. These eigenfrequencies do not change even when the number of layers is changed, hence the two eigenfrequencies can be considered as those corresponding to eigenmodes which have large amplitude in input and output domains while small amplitude in connection part.

4. Conclusion

The present study was undertaken to develop a method for investigations of one directional 2D finite periodic bounded acoustic structures. This was achieved by calculation of transmissivity by solving size-reduced system matrix and bandgap by using BEM combined with the block SS method. By using Bloch’s theorem to the unit cell, the block SS method has been directly applied to computation of the NEP formulated by BEM. The fictitious eigen
values observed in the computation could be removed by employing Burton-Miller’s method. Furthermore, the eigenfrequencies of the finite periodic bounded structure was also obtained and they showed a bandgap which had a good agreement with the results obtained by computing for the unit cell.

The present work is being further developed to solve the size-reduced matrix by using recursive matrix, because this matrix can be divided into different parts, among which a simple diagonal band matrix is found.

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References


