Transactions of JASCOME Vol. 13 (Nov. 2013), Paper No. 19-131129 JASCOME

Topology optimization for 2D heat conduction problems using boundary element method and level set method

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The heat conduction problem is one of the typical physical problems and shape and/or topology optimization considering the heat property is quite important. In this paper, a level set based topology optimization method with the boundary element method (BEM) for 2D heat conduction problems is investigated. The boundary element method is convenient in generating the boundary mesh at every iterative step of optimization. The effectiveness of the proposed approach is demonstrated through some numerical examples. From these examples, we conclude that the optimization for heat problem can be dealt with the boundary element method.

Key Words: Topology optimization, Level set method, Heat conduction problem, Boundary element method

1. Introduction

This paper discusses a design method for heat conduction problems. The heat conduction problem is one of the typical physical problems. A method to obtain reasonable structure (less material and high heat efficiency) for heat devices is desired in the field of mechanical engineering. There are some optimization method to design material shapes, for example, size [(1)], shape [(2)] and topology optimization [(3)]. We here employ the topology optimization method to design the reasonable structure.

The topology optimization is widely studied and applied to a variety of structural optimization problems such as the stiffness maximization problem [(3, 4)], compliant mechanisms [(5)], and thermal problems [(6)].

Since the topology optimization concerns with material distribution, characteristic functions are employed to distinguish the material region from non-material region (void domain). However, this always results in the discontinuity of the material distribution. To remove the discontinuity, many regularization techniques are proposed, such as homogenization design method [(3)], and solid isotropic material with penalization (SIMP) method [(7)] for the relaxation of the design domain.

These methods play an important role in eliminating the discontinuity of the obtained optimum shape, however, they usually give an "intermediate material", which is neither material nor non-material domain. To resolve this issue, Wang et al. proposed a level set based structural optimization method. It is reviewed as a new kind of method, which is based on solving the Hamilton-Jacobi partial differential equation, with an appropriate normal velocity moving boundary normal to the interface. This method can obtain the optimum shape with the intermediate material but its region is narrow.

Addition to the intermediate domain, the obtained shapes are usually not smooth. Yamada et al. proposed a topology optimization method using a level set method incorporating with a fictitious interface energy [(8)] derived from the phase field concept, in which a fictitious interface energy term is used [(9)] to regularize the optimization problem.

Most of the above optimization methods use the finite element method (FEM) as a solver. When the FEM is used, the cost for creating mesh at every step of the optimization is expensive. To overcome this issue, we present a level set based topology optimization for 2D heat conduction problem using boundary element method (BEM) [(10)] in this study. When we use the BEM, we may discretize the boundary of the material, which enables us to save the numerical cost for creating mesh. To this end, we combine the method proposed by Yamada et al.[(8)], and try to develop the basic algorithm

Received Sep. 16, 2013, accepted Oct. 22, 2013

of topology optimization approach using BEM.

2. Formulation

2.1. The concept of level set method

The level set method (LSM) is an effective numerical technique to represent interfaces and shapes of domain. The level set function ϕ is defined as follows:

$$\begin{cases} \phi(\mathbf{x}) > 0, & \mathbf{x} \in \Omega \\ \phi(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma \\ \phi(\mathbf{x}) < 0, & \mathbf{x} \in D \backslash \Omega, \end{cases} \tag{1}$$

where D, Ω and Γ denote the fixed design domain which include material domain and non-material domain, the material domain and the boundary between material and non-material domain, respectively. We also define a characteristic function χ. The characteristic

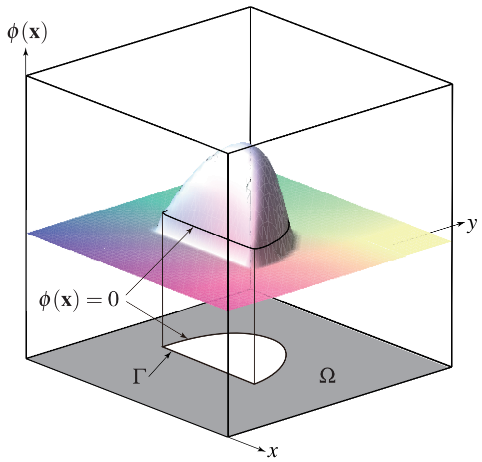

Fig. 1 Fixed design domain

function is associated with the level set function $\phi(x)$ as follows:

$$\chi_\phi(x) = \begin{cases} 1, & \phi(x) \geq 0 \\ 0, & \phi(x) < 0. \end{cases} \tag{2}$$

2.2. Topology optimization problem

We consider the following optimization problem where the objective function is defined only on boundary:

$$\inf_{\phi} F(\chi_\phi) = \int_\Gamma f(u,q)\, d\Gamma, \tag{3}$$

subject to

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = \bar{u} & \text{on } \Gamma_u \\ q = \bar{q} & \text{on } \Gamma_q \end{cases} \tag{4}$$

and

$$G(\chi_\phi) = \int_D \chi_\phi \, d\Omega - G_{\max} \leq 0, \tag{5}$$

where $f(u,q)$ is a function of u and q defined on Γ or part of Γ, u and $q = -k\dfrac{\partial u}{\partial n}$ are temperature and heat flux of the temperature field with the thermal conductivity k, respectively. Also, n is an outward normal vector on Γ. $G_{\max}$ is the admissible upper limit of the area of the material domain Ω.

Using Lagrange's method, the optimization problem (3)–(5) is turned to be the following unconstrained optimization problem:

$$\inf_{\phi} \overline{F} = F(\chi) + I + \lambda G, \tag{6}$$

where λ is the Lagrange multiplier for equation (5) and I is a function defined as follows:

$$I = \int_D \mu(\nabla^2 u)\, d\Omega = 0. \tag{7}$$

The above optimization problem has to satisfy the following Karush-Kuhn-Tucker (KKT) [11] conditions:

$$F' + I' + \lambda = 0, \quad \lambda G = 0, \quad \lambda \geq 0, \quad G \leq 0, \tag{8}$$

where a prime symbol ($'$) denotes a topological derivative, which characterizes a sensitivity of the objective function (3) when a infinitely small circular hole is created. By solving the KKT conditions, we can obtain an optimum distribution of the level set function and the associated shape and topology of Ω. If we solve inequality (8) as is, however, we obtain the level set function which might be discontinuous at everywhere. This issue is known to be caused by the ill-posedness of the topology optimization. To avoid this, we here use the Tikhonov regularization [12], i.e., we replace the objective function $\overline{F}$ (Eq. (8)) by the following $\overline{F}_\mathrm{R}$:

$$\overline{F}_R = \overline{F} + \frac{1}{2}\tau \int_D |\nabla\phi|^2 \, d\Omega, \tag{9}$$

where τ is a regularization parameter. We can control the complexity of the obtained optimal configuration by adjusting τ.

Since it is difficult to obtain the level set function satisfied the KKT conditions, we introduce a fictitious time t and assume that the time derivative of the level set function ϕ is proportional to the topological derivative of $\overline{F}_R$ as follows:

$$\frac{\partial \phi}{\partial t} = -K\overline{F}_R' \ \text{ in } D. \tag{10}$$

As a result, an optimum distribution of the level set function can be obtained as the solution of the following boundary value problem:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} = -K(F' + I' + \lambda - \tau\nabla^2\phi) & \text{in } D \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on } \Gamma \backslash \Gamma_N \\ \phi = 1 & \text{on } \Gamma_N, \end{cases} \tag{11}$$

where K is a constant. Γ_N is the non-design boundary. $F' + I'$ can be derived to be $-2k\nabla u \cdot \nabla \mu$ by introducing an adjoint field [13]. The boundary value problem of adjoint field μ is governed by the following boundary value problem:

$$\begin{cases} \nabla^2 \mu = 0 & \text{in } \Omega \\ \mu = -\dfrac{\partial f}{\partial q} & \text{on } \Gamma_u \\ -k\dfrac{\partial \mu}{\partial n} = \dfrac{\partial f}{\partial u} & \text{on } \Gamma_q. \end{cases} \tag{12}$$

We here use the finite element method to solve the boundary value problem Eq. (11).

2.3. Boundary element method

We use the boundary element method to obtain u and μ in equations (4) and (12). For the boundary value problem of temperature field u in equation (4), in the case of the boundary Γ is smooth, the boundary integral equation is written as follows:

$$\frac{1}{2}u(y) = \int_\Gamma u^*(x,y)q(x)d\Gamma(x) - \int_\Gamma q^*(x,y)u(x)d\Gamma(x), \quad y \in \Gamma, \tag{13}$$

where $u^*(x,y)$ and $q^*(x,y)$ are the fundamental solution of Laplace's equation and its normal derivative, respectively. $u^*(x,y)$ is given as follows:

$$u^*(x,y) = \frac{1}{2\pi}\ln\frac{1}{r}, \tag{14}$$

$$q^*(x,y) = \frac{-1}{2\pi r}\frac{\partial r}{\partial n}, \tag{15}$$

where r denotes the distance between x and y. By discretizing equation (13), we have the following system of equations:

$$[H]\{u\} = [G]\{q\}. \tag{16}$$

After moving the unknowns to the left-hand side and the knowns to the right-hand side, we have

$$[A]\{X\} = \{Y\}, \tag{17}$$

where $\{X\}$ is the vector consisting of only unknown nodal values and $\{Y\}$ is the vector obtained by multiplying the known nodal values with corresponding parts of the coefficient matrices [H] and [G]. Once we obtain the solution of equation (17), the temperature field u inside the domain Ω can be calculated by the following integral representation:

$$u(y) = \int_\Gamma u^*(x,y)q(x)d\Gamma(x) - \int_\Gamma q^*(x,y)u(x)d\Gamma(x), \quad y \in \Omega. \tag{18}$$

We can obtain ∇u by differentiating equation (18). The adjoint filed μ can be obtained in the exact same manner.

3. Numerical Examples

The numerical experiment are run on a computer with Intel Core i7-2600 processor whose clock rate is 3.40GHz.

3.1. Numerical Example 1

To check the effectiveness of the proposed methodology for 2D heat conduction problem, we consider a heat conduction problem shown as Fig. 2.

The size of fixed design domain D is set to be 0.5 m $\times$ 0.5 m. As an initial configuration, we filled the fixed design domain with steel whose thermal conductivity is 17.0 $\mathrm{W/(m \cdot K)}$ or with steel whose thermal conductivity is 7.0 $\mathrm{W/(m \cdot K)}$.

The prescribed temperature and heat flux are given $\bar{u} = 100°\mathrm{C}$ on Γ_u and $\bar{q} = 1000\ \mathrm{W/m}$ on Γ_q, respectively, where Γ_u and Γ_q are defined as shown in Fig. 2. The length of Γ_u and Γ_q are 0.05 m. The rest of the boundary is insulate boundary.

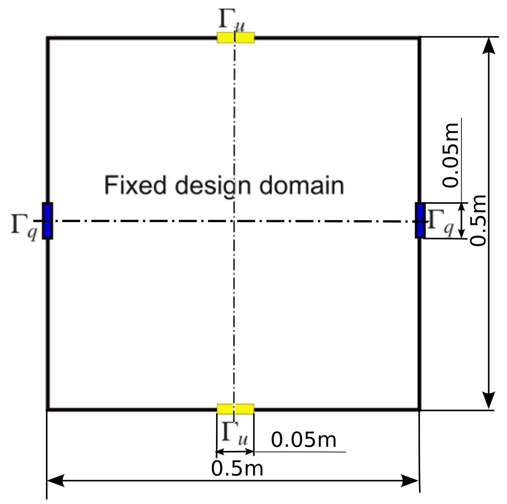

Fig. 2 Fixed design domain for numerical example 1.

Our objective is to minimize the objective function defined on Γ_q. The objective function is given as follows:

$$F(\chi_\phi) = \int_{\Gamma_q} (u - \hat{u})^2\, d\Gamma, \tag{19}$$

where $\hat{u}$ is the objective temperature, which is set to be 60°C. $G_{\max}$ in Eq. (5) is set to 20% of the fixed design domain. The regularization parameter is set as $\tau = 3.0 \times 10^{-3}$. The coefficient K of the time evolution equation in Eq. (10) is set as 5.0. The time increment δt is set as 0.1. We use square shaped meshes to solve the boundary value problem (Eq. (11)). The fixed design domain is divided into 60 $\times$ 60 cells. The boundary element meshes to solve the boundary value problems (4) and (12) are created by tracking the zeros of the value of the level set function which is evaluated on the finite element nodes.

The obtained optimum distributions of material are shown as Fig. 3 and Fig. 4. The computational times are 1012sec for both cases with different thermal conductivities.

From Fig. 3 and Fig. 4, for these problems, the thermal conductivity is irrelevant to the optimal shape. Also, we can see that the obtained shape has smooth boundaries. The objective function values against the iterative step of optimization are shown in Fig. 5 and Fig. 6. From these two figures, we can find that the objective functions decrease step by step and converge to about 0.60 for the case with thermal conductivity as 17.0 $\mathrm{W/(m \cdot K)}$ and 0.10 with thermal conductivity as 7.0 $\mathrm{W/(m \cdot K)}$.

Next, we investigate the influence of the initial configuration to the optimum results. We here use steel whose conductivity is 17.0

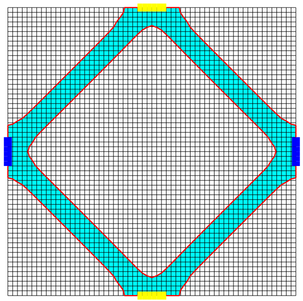

Fig. 3 Obtained shapes for the numerical example 1 with thermal conductivity as 17.0 W/(m·K)..

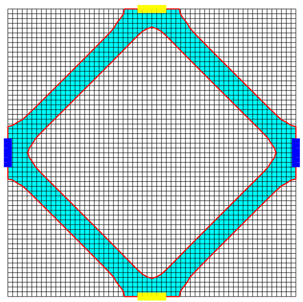

Fig. 4 Obtained shapes for the numerical example 1 with thermal conductivity as 7.0 W/(m·K).

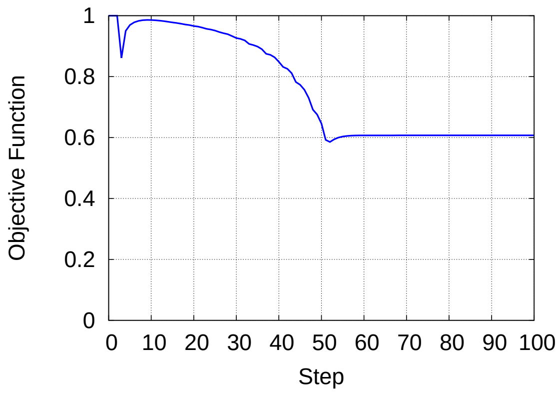

Fig. 5 Normalized objective function value history for the numerical example 1 with thermal conductivity as 17.0 W/(m·K).

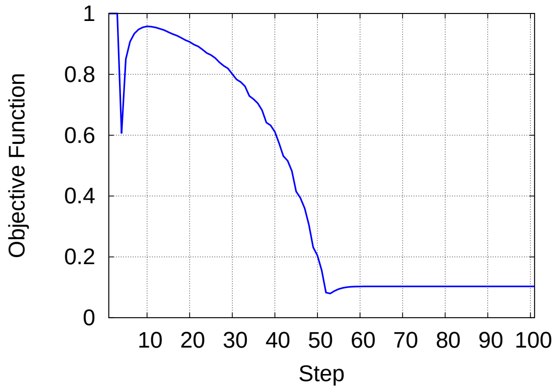

Fig. 6 Normalized objective function value history for the numerical example 1 with thermal conductivity as 7.0 W/(m·K).

W/(m·K) as material. We consider the following 3 initial configurations; (1) the whole fixed design domain is filled with Ω, (2) Ω has 4 holes and (3) Ω has 9 holes. We henceforth denote the above initial configurations as "without hole", "with 4 holes" and "with 9 holes", respectively. Figs. 7–9 show initial, intermediate and optimum configurations. For these 3 cases, the regularization parameter is set as $\tau = 5.0 \times 10^{-3}$, the coefficient K of the time evolution equation in Eq. (10) is set as 5.0, and the time increment δt is set as 0.1. From these figures we found that the initial configuration has influences on the obtained optimum shape. This is partly because that the topological derivative $F' + I'$ in Eq. (11) is equal to zero in $D \setminus \Omega$. The computational time is 316 sec for the case without holes, 517 sec for the case with 4 holes and 546 sec for the case with 9 holes.

Fig. 10 shows the objective function history for each initial configuration. We can confirm that the objective function for all initial configurations converged to similar values. With these observations, we conclude that the problem treated in this section has similar solutions.

3.2. Numerical Example 2

We next consider the problem shown as Fig.11. The prescribed temperature $\bar{u} = 100°\text{C}$ on Γ_u, and the prescribed heat flux $\bar{q} = 1000\text{W/m}$ on Γ_q. The rest of the boundary is insulate boundary.

The objective function for this example is same as that for the previous one in Eq. (19) with $\hat{u} = 60°\text{C}$. $G_{\max}$ in Eq. (5) is set to 20% of the fixed design domain. The regularization parameter is set as $\tau = 5.0 \times 10^{-3}$. The coefficient K of time evolution equation (11) is set as 5.0 and time increment δt is set as 0.1.

The boundary element is generated by searching 0-value level-set function based on the mesh of 40×40 cells. The obtained topology distributions of different initial configurations are shown as Figs. 12, 13 and 14. Also, the objective function of this example with different initial configurations are shown as Fig. (15). From these figures, we conclude the following:

- The objective function decreased enough.
- The obtained configurations have smooth boundary.
- When the initial configuration is different, the proposed method could give the similar optimum configuration.

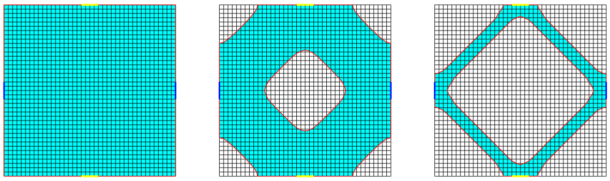

Fig. 7 (left:) Initial (center:) intermediate (right:) an optimal configuration “without hole” for the numerical example 1.

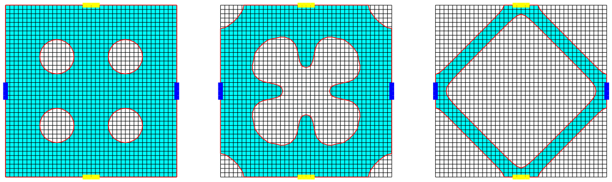

Fig. 8 (left:) Initial (center:) intermediate (right:) an optimal configuration “with 4 holes” for the numerical example 1.

The computational time is 344 sec for the case without holes, 382 sec for the case with 4 holes and 391 sec for the case with 9 holes.

4. Conclusion

In this paper, we proposed a new topology optimization method combined with the level set method and the boundary element method for 2D heat conduction problem. Through the numerical examples, we have proved that the proposed methodology is effective for 2D heat conduction problems. Besides, we have conclude that cases of different initial configurations have the same converging tendency, and converged to similar shapes. In particular, we have successfully obtained a few material smooth shapes with specified temperatures on a part of the material boundary. We also note that the computational time is short for all the cases. In the following work, we will move our research more on the effect of thermal conductivity to the convergency and later investigate topology optimization using BEM for acoustic and electromagnetic problems.

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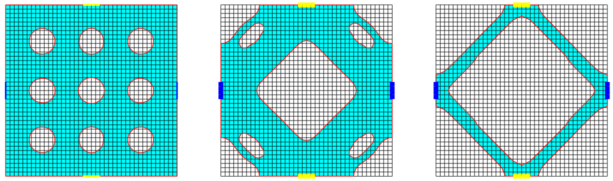

Fig. 9 (left:) Initial (center:) intermediate (right:) an optimal configuration “with 9 holes” for the numerical example 1.

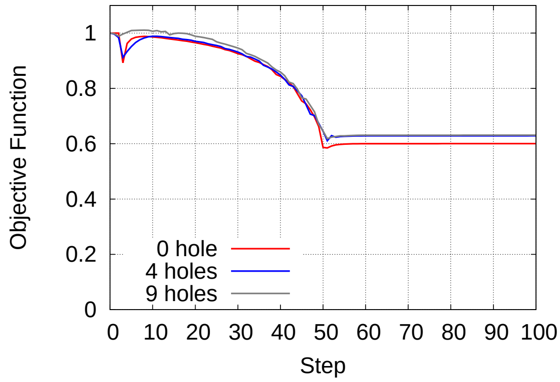

Fig. 10 Normalized objective function value history.

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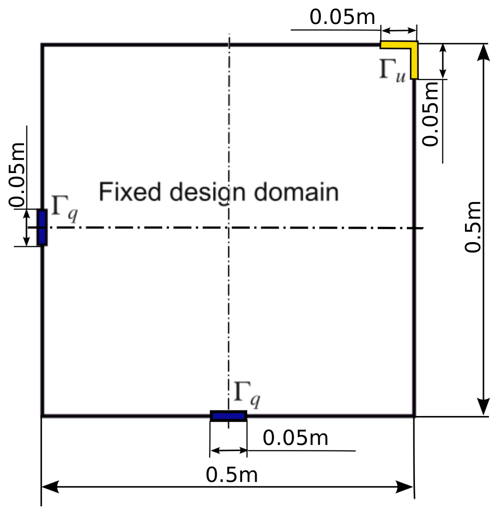

Fig. 11 Fixed design domain for the numerical example 2.

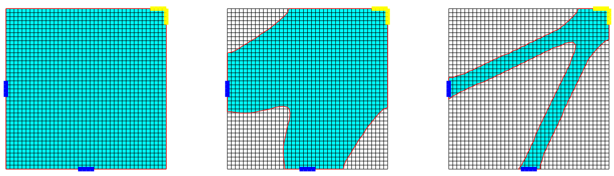

Fig. 12 (left:) Initial (center:) intermediate (right:) an optimal configuration "without hole" for numerical example 2.

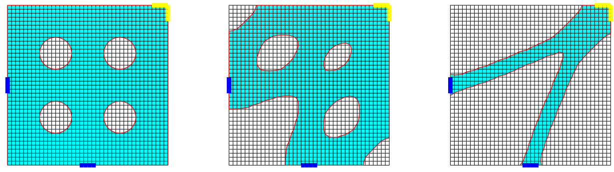

Fig. 13 (left:) Initial (center:) intermediate (right:) an optimal configuration "with 4 holes" for numerical example 2.

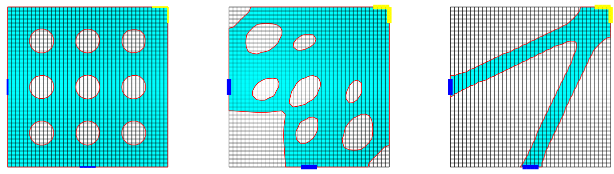

Fig. 14 (left:) Initial (center:) intermediate (right:) an optimal configuration "with 9 holes" for numerical example 2.

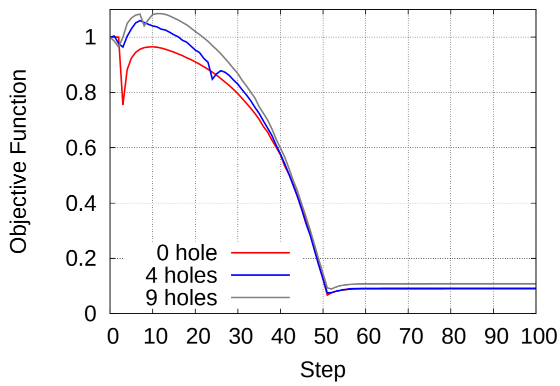

Fig. 15 Objective function value history for the numerical example 2.

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