LAPLACE 方程式の不足決定系問題における随伴直接解法

DIRECT VARIATIONAL METHOD FOR SOLUTION OF UNDER-DETERMINED PROBLEM OF THE LAPLACE EQUATION BY BEM

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The Laplace equation in two dimensions is considered in a domain enclosed by smooth curve. Dirichlet data are prescribed on a part of the boundary, while Neumann data are so prescribed on the other part of the boundary that some part of the boundary is left with no prescribed boundary data. This kind of under-determined problem is not uniquely solvable, and one of the solutions is defined in terms of a variational problem. As the necessary condition for a functional to attain the minimum, primary and adjoint boundary value problems of the Laplace equation are derived. A direct method of solution using the BEM is presented. Simple examples are shown to demonstrate the validity of our treatment.

Key Words: *Inverse problem, Boundary value identification, Direct method*

1 Introduction

Let Ω be a simply connected bounded domain enclosed by a smooth boundary Γ in $\boldsymbol{R}^2$. Let $\boldsymbol{n}$ be the exterior unit normal to the boundary.

We consider the Laplace equation;

$$-\Delta u(\boldsymbol{x}) = 0\,, \quad \boldsymbol{x} \in \Omega \tag{1}$$

subject to Dirichlet and Neumann data;

$$u|_{\Gamma_u} = \bar{u} \quad \text{and} \quad \frac{\partial u}{\partial n} = q|_{\Gamma_q} = \bar{q} \tag{2}$$

given on respective non-zero measure parts of the boundary Γ_u and Γ_q. Here we notice that the components Γ_u and Γ_q can be taken arbitrarily as long as $\Gamma_u \cap \Gamma_q = \phi$ and $\Gamma_u \cup \Gamma_q \neq \Gamma$. This problem is called under-determined problem, and it can be understood as an identification problem of boundary values in the context of inverse problems.

We can interpret our under-determined problem as an exterior field problem in electro-statics by embedding the domain Ω into the whole domain $\boldsymbol{R}^2$. We regard the boundary components Γ_u and Γ_q as sheet electrodes. Electric potential $\bar{u}$ is prescribed on Γ_u, and electric current $\bar{q}$ is prescribed on Γ_q. Then the electro-static field $u(\boldsymbol{x})$ induced by the charge on the electrodes can be expressed by the sum of two single layered potentials as follows:

$$u(\boldsymbol{x}) = \int_{\Gamma_u} G(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}) + \int_{\Gamma_q} G(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y})\,, \qquad \boldsymbol{x} \in \boldsymbol{R}^2 \setminus (\Gamma_u \cup \Gamma_q)$$

where $G(\boldsymbol{x};\boldsymbol{y})$ is the fundamental solution to the Laplacian;

$$-\Delta G(\boldsymbol{x};\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y}) \tag{3}$$

with the Dirac measure δ at the point $\boldsymbol{y}$. In two dimensions we know

$$G(\boldsymbol{x};\boldsymbol{y}) = \frac{1}{2\pi} \ln \frac{1}{\|\, \boldsymbol{x}-\boldsymbol{y} \,\|}\,. \tag{4}$$

The continuous function $\sigma(\boldsymbol{y})$ denotes the density of electric charge on $\Gamma_u \cup \Gamma_q$.

Since the single layered potential is continuous everywhere in $\boldsymbol{R}^2$, we have

$$u(\boldsymbol{x}) = \int_{\Gamma_u} G(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}) + \int_{\Gamma_q} G(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \Gamma_u .$$

On the other hand, since the double layered potential is discontinuous on the boundary Γ_q and it satisfies the jump relation, we have

$$\frac{\partial u}{\partial n}(\boldsymbol{x}) = \int_{\Gamma_u} \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}) - \frac{1}{2}\sigma(\boldsymbol{x}) + \int_{\Gamma_q} \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \qquad \boldsymbol{x} \in \Gamma_q$$

for the smooth boundary.

From the boundary conditions eqn (2) we see that the density function must satisfy the following system of boundary integral equations.

$$\int_{\Gamma_u} G(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}) + \int_{\Gamma_q} G(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}) = \bar{u}(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Gamma_u$$

and

$$-\frac{1}{2}\sigma(\boldsymbol{x}) + \int_{\Gamma_u} \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}) + \int_{\Gamma_q} \frac{\partial G}{\partial n(\boldsymbol{x})}(\boldsymbol{x};\boldsymbol{y})\sigma(\boldsymbol{y})d\Gamma(\boldsymbol{y}) = \bar{q}(\boldsymbol{x}) \qquad \boldsymbol{x} \in \Gamma_q .$$

This discussion leads to the indirect boundary element method for an approximate solution of the under-determined problem.

In the direct boundary element method we start with the fact that the boundary values $u|_\Gamma$ and $q|_\Gamma$ should satisfy the boundary integral equation;

$$\frac{1}{2}u(\boldsymbol{x}) + \int_\Gamma \frac{\partial G}{\partial n}(\boldsymbol{x};\boldsymbol{y})u(\boldsymbol{y})d\Gamma(\boldsymbol{y}) = \int_\Gamma G(\boldsymbol{x};\boldsymbol{y})q(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma. \quad (5)$$

In the preceding paper(1), the authors presented an idea for the direct variational method by paraphrasing the inverse boundary value problems of the Laplace equation into primary and adjoint problems. In the present paper, we are concerned only with the under-determined problem, and we consider the direct boundary element method by showing some numerical examples.

2 Variational Problem

Let Γ_u^c and Γ_q^c be complement sets of Γ_u and Γ_q, respectively. We recast the problem eqns (1), (2) into the following variational problem: Find $u|_{\Gamma_u^c} = \omega$ that minimizes the functional

$$J(\omega) = \int_{\Gamma_q} |q(\boldsymbol{x};\omega) - \bar{q}(\boldsymbol{x})|^2 d\Gamma(\boldsymbol{x}) \qquad (6)$$

subject to

$$-\Delta u(\boldsymbol{x};\omega) = 0, \quad \boldsymbol{x} \in \Omega \qquad (7)$$

$$u|_{\Gamma_u} = \bar{u} \quad \text{and} \quad u|_{\Gamma_u^c} = \omega . \qquad (8)$$

The first variation of the functional eqn (6) is given in the form(2);

$$J'(\omega)|_{\Gamma_u^c} = \frac{\partial v}{\partial n}(\boldsymbol{x};\omega), \qquad (9)$$

where $v(\boldsymbol{x};\omega)$ is the solution of the following adjoint problem to the primary problem eqns (7), (8);

$$-\Delta v(\boldsymbol{x};\omega) = 0, \quad \boldsymbol{x} \in \Omega \qquad (10)$$

$$v|_{\Gamma_q^c} = 0 \quad \text{and} \quad v|_{\Gamma_q} = 2\{q(\boldsymbol{x};\omega) - \bar{q}(\boldsymbol{x})\} . \qquad (11)$$

The functional $J(\omega)$ attains its minimum $J(\omega) = 0$ at any solution $u(\boldsymbol{x})$ of the problem eqns (1), (2). Conversely the necessary condition

$$J'(\omega)|_{\Gamma_u^c} = \frac{\partial v}{\partial n}|_{\Gamma_u^c} = 0 \qquad (12)$$

for $J(\omega)$ to attain the minimum yields the trivial solution $v(\boldsymbol{x};\omega) = 0$ of the adjoint problem eqns (10), (11). This implies $q|_{\Gamma_q} = \bar{q}$, which corresponds a solution $u(\boldsymbol{x})$ of the problem eqns (1), (2). Therefore our variational problem consisting of the minimization of $J(\omega)$ is equivalent to the problem eqns (1), (2).

3 Boundary Element Method

We divide the whole boundary Γ into the series of n boundary elements as $\Gamma \simeq \Gamma^h = \cup_{j=1}^n \Gamma_j$ for its approximation, where h stands for some representative size of the boundary elements. Here the boundary element subdivision should be in accordance with the boundary components Γ_u and Γ_q.

We approximate the boundary values $u|_\Gamma$ and $q|_\Gamma$ by introducing the interpolation functions $N_j(\boldsymbol{x})$ in the form;

$$u|_\Gamma \simeq u^h(\boldsymbol{x}) = \sum_{j=1}^n N_j(\boldsymbol{x})u_j , \qquad (13)$$

$$q|_\Gamma \simeq q^h(\boldsymbol{x}) = \sum_{j=1}^n N_j(\boldsymbol{x})q_j , \quad \boldsymbol{x} \in \Gamma \qquad (14)$$

with approximate nodal values u_j and q_j to the exact nodal values $u(\boldsymbol{x}_j)$ and $q(\boldsymbol{x}_j)$, respectively, at the nodes $\boldsymbol{x}_j$ $(j = 1, 2, \cdots, n)$ on the boundary Γ. We approximate the boundary values $v|_\Gamma$ and $r|_\Gamma = \dfrac{\partial v}{\partial n}$ also in the form;

$$v|_\Gamma \simeq v^h(\boldsymbol{x}) = \sum_{j=1}^{n} N_j(\boldsymbol{x}) v_j\,, \tag{15}$$

$$r|_\Gamma \simeq r^h(\boldsymbol{x}) = \sum_{j=1}^{n} N_j(\boldsymbol{x}) r_j\,, \quad \boldsymbol{x} \in \Gamma \tag{16}$$

with approximate nodal values v_j and r_j to the exact $v(\boldsymbol{x}_j)$ and $r(\boldsymbol{x}_j)$, respectively, at $\boldsymbol{x}_j$ on Γ. We take those n nodes $\boldsymbol{x}_j$ as collocation points in order to fully discretize the boundary integral equations.

We apply this discretization procedure to the boundary integral equation (5) corresponding to the primary problem eqns (7), (8), which results in the system of linear equations in the matrix form;

$$[H]\{u\} = [G]\{q\}\,. \tag{17}$$

We apply the procedure again to a boundary integral equation that corresponds to the adjoint problem eqns (10), (11) to obtain

$$[H]\{v\} = [G]\{r\} \tag{18}$$

with the same $n \times n$ coefficient matrices $[H]$ and $[G]$.

We denote by n_1 the number of nodes on Γ_u, and by n_2 the number of nodes on Γ_q, respectively. Let $n_1^c = n - n_1$ and $n_2^c = n - n_2$, being the respective numbers of nodes on Γ_u^c and Γ_q^c. According to the respective boundary components Γ_u and Γ_q we can write the column vectors $\{u\}$ and $\{q\}$ in the form;

$$\{u\} = \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}, \quad \{q\} = \left\{ \begin{array}{c} q_1 \\ q_2 \end{array} \right\},$$

where the n_1 nodal values u_j on Γ_u are collected in $\{u_1\}$, and the n_1^c nodal values on Γ_u^c in $\{u_2\}$, whereas the n_2 nodal values q_j on Γ_q are collected in $\{q_2\}$, and the n_2^c nodal values on Γ_q^c in $\{q_1\}$. We may write this partition in a more explanatory way as

$$\begin{array}{l} \{u\} = \left\{ \begin{array}{ccc} u_1 & \text{on} & \Gamma_u \\ u_2 & \text{on} & \Gamma_u^c \end{array} \right\} \begin{array}{l} \}n_1 \\ \}n_1^c \end{array}, \\ \{q\} = \left\{ \begin{array}{ccc} q_1 & \text{on} & \Gamma_q^c \\ q_2 & \text{on} & \Gamma_q \end{array} \right\} \begin{array}{l} \}n_2^c \\ \}n_2 \end{array}, \end{array} \tag{19}$$

where the dimension of each column vector is indicated.

In the similar way we write

$$\begin{array}{l} \{v\} = \left\{ \begin{array}{ccc} v_1 & \text{on} & \Gamma_q^c \\ v_2 & \text{on} & \Gamma_q \end{array} \right\} \begin{array}{l} \}n_2^c \\ \}n_2 \end{array}, \\ \{r\} = \left\{ \begin{array}{ccc} r_1 & \text{on} & \Gamma_u \\ r_2 & \text{on} & \Gamma_u^c \end{array} \right\} \begin{array}{l} \}n_1 \\ \}n_1^c \end{array}. \end{array} \tag{20}$$

Then the systems eqns (17) and (18) can be written respectively in the partitioned form;

$$\Big[\overbrace{H_1^{(1)}}^{n_1} \;\; \overbrace{H_2^{(1)}}^{n_1^c} \Big] \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\} = \Big[\overbrace{G_1^{(1)}}^{n_2^c} \;\; \overbrace{G_2^{(1)}}^{n_2} \Big] \left\{ \begin{array}{c} q_1 \\ q_2 \end{array} \right\} \tag{21}$$

and

$$\Big[\overbrace{H_1^{(2)}}^{n_2^c} \;\; \overbrace{H_2^{(2)}}^{n_2} \Big] \left\{ \begin{array}{c} v_1 \\ v_2 \end{array} \right\} = \Big[\overbrace{G_1^{(2)}}^{n_1} \;\; \overbrace{G_2^{(2)}}^{n_1^c} \Big] \left\{ \begin{array}{c} r_1 \\ r_2 \end{array} \right\}, \tag{22}$$

where numbers of columns of the coefficient matrices are indicated.

Before discussing a direct method for the solution of linear systems of equations (21) and (22), we outline the indirect boundary element method. We divide the boundary component Γ_u into the series of n_1 boundary elements as $\Gamma_u \simeq \Gamma_u^h = \bigcup_{j=1}^{n_1} \Gamma_j$ for its approximation. Similarly we divide $\Gamma_q \simeq \bigcup_{j=n_1+1}^{n_1+n_2} \Gamma_j$. We shall consider constant boundary elements for simplicity by taking nodes $\boldsymbol{x}_j$ at the middle of each Γ_j.

The density $\sigma(\boldsymbol{x})$ on $\Gamma_u \cap \Gamma_q$ is approximated in the form:

$$\sigma|_{\Gamma_u \cup \Gamma_q} \simeq \sigma^h(\boldsymbol{x}) = \sum_{j=1}^{n_1+n_2} \sigma_j \chi_j(\boldsymbol{x})$$

with approximate nodal values σ_j to the exact $\sigma(\boldsymbol{x}_j)$ and the characteristic set functions $\chi_j(\boldsymbol{x})$ for the element Γ_j;

$$\chi_j(\boldsymbol{x}) = \left\{ \begin{array}{ll} 1 & (\boldsymbol{x} \in \Gamma_j) \\ 0 & (\boldsymbol{x} \notin \Gamma_j). \end{array} \right.$$

By using the approximation σ^h in place of the exact σ, the method of collocations in which the system of boundary integral equations are collocated at the nodes $\boldsymbol{x}_i$ $(i = 1, 2, \ldots, n_1 + n_2)$ yields

$$\sum_{j=1}^{n_1} \int_{\Gamma_j} G(\boldsymbol{x}_i; \boldsymbol{y})\, d\Gamma(\boldsymbol{y}) \sigma_j + \sum_{j=n_1+1}^{n_1+n_2} \int_{\Gamma_j} G(\boldsymbol{x}_i; \boldsymbol{y})\, d\Gamma(\boldsymbol{y}) \sigma_j = \overline{u}(\boldsymbol{x}_i) \quad (i = 1, 2, \ldots, n_1)$$

and

$$-\frac{1}{2}\sigma_i+\sum_{j=1}^{n_1}\int_{\Gamma_j}\frac{\partial G}{\partial n(x)}(x_i;y)\,d\Gamma(y)\sigma_j$$
$$+\sum_{j=n_1+1}^{n_1+n_2}\int_{\Gamma_j}\frac{\partial G}{\partial n(x)}(x_i;y)\,d\Gamma(y)\sigma_j=\bar{q}(x_i)$$
$$(i=n_1+1,n_1+2,\ldots,n_1+n_2)$$

We solve this linear system of equations for n_1+n_2 unknowns σ_j. The approximate electro-static field $u^h(x)$ to the exact $u(x)$ at $x\in R^2\setminus(\Gamma_u\cup\Gamma_q)$ can be obtained by the formula;

$$u^h(x)=\sum_{j=1}^{n_1}\int_{\Gamma_j}G(x;y)\,d\Gamma(y)\sigma_j$$
$$+\sum_{j=n_1+1}^{n_1+n_2}\int_{\Gamma_j}G(x;y)\,d\Gamma(y)\sigma_j.$$

4 Direct Method of Solution

We insert boundary conditions of primary and adjoint problems into the partitioned systems of eqns (21), (22): From eqn (8) we have

$$\{u_1\}=\{\bar{u}_1\},\quad \{u_2\}=\{\omega\}. \qquad (23)$$

From eqn (11) we have

$$\{v_1\}=\{0\},\quad \{v_2\}=2(\{q_2\}-\{\bar{q}_2\}), \qquad (24)$$

and from eqn (12) we have

$$\{r_2\}=\{0\}. \qquad (25)$$

Therefore the systems eqns (21), (22) are reduced to the form;

$$\begin{bmatrix} H_1^{(1)} & H_2^{(1)} \end{bmatrix}\begin{Bmatrix}\bar{u}_1\\ \omega\end{Bmatrix}=\begin{bmatrix} G_1^{(1)} & G_2^{(1)} \end{bmatrix}\begin{Bmatrix}q_1\\ \frac{1}{2}v_2+\bar{q}_2\end{Bmatrix} \qquad (26)$$

and

$$\begin{bmatrix} H_1^{(2)} & H_2^{(2)} \end{bmatrix}\begin{Bmatrix}0\\ v_2\end{Bmatrix}=\begin{bmatrix} G_1^{(2)} & G_2^{(2)} \end{bmatrix}\begin{Bmatrix}r_1\\ 0\end{Bmatrix}, \qquad (27)$$

respectively.

We combine eqns (26) and (27). We take unknown nodal values to the left of the equation to obtain

$$\begin{bmatrix} \overbrace{-G_1^{(1)}}^{n_2^c} & \overbrace{H_2^{(1)}}^{n_1^c} & \overbrace{O}^{n_1} & \overbrace{-\frac{1}{2}G_2^{(1)}}^{n_2} \\ O & O & -G_1^{(2)} & H_2^{(2)} \end{bmatrix}\begin{Bmatrix}q_1\\ \omega\\ r_1\\ v_2\end{Bmatrix}=\begin{bmatrix} \overbrace{-H_1^{(1)}}^{n_1} & \overbrace{G_2^{(1)}}^{n_2} \\ O & O \end{bmatrix}\begin{Bmatrix}\bar{u}_1\\ \bar{q}_2\end{Bmatrix}. \qquad (28)$$

We notice that the coefficient matrix on the left hand side of the augmented new system of linear eqns (28) is square of order $2n$.

5 Numerical Example

Suppose that the harmonic function

$$u(x_1,x_2)=x_1^2-x_2^2=r^2\cos(2\vartheta) \qquad (29)$$

in two dimensions with the polar coordinates $x_1=r\cos\vartheta$, $x_2=r\sin\vartheta$ serves as a solution of the under-determined problem eqns (1), (2) in the unit circle

$$\Omega=\left\{(r,\vartheta)\ \middle|\ 0\le r=\sqrt{x_1^2+x_2^2}<1,\ 0\le\vartheta<2\pi\right\} \qquad (30)$$

with the Dirichlet data

$$\bar{u}=\cos(2\vartheta)\quad\text{on}\quad\Gamma_u=\{(1,\vartheta)\ |\ 0\le\vartheta\le\pi/2\}$$

and the Neumann data

$$\bar{q}=2\cos(2\vartheta)\quad\text{on}\quad\Gamma_q=\{(1,\vartheta)\ |\ \pi<\vartheta<3\pi/2\}$$

as shown in Fig. 1.

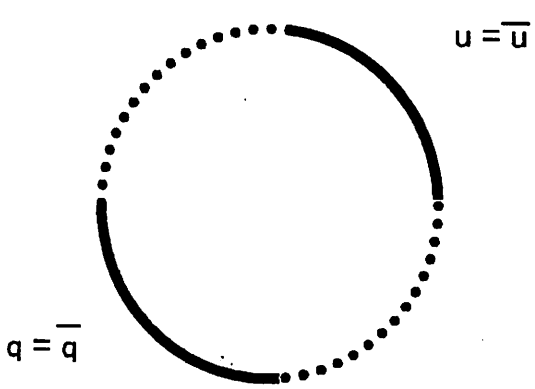
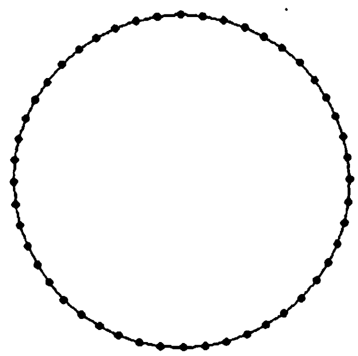


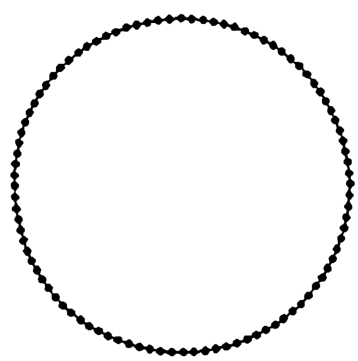
Fig. 1 Under-determined problem

The collocation boundary element method is used. The u- and v-values along the boundary are approximated by using C^0 linear elements. The q- and r-values along the boundary are also approximated by using C^0 linear elements. However at the edge of the boundary components Γ_u and Γ_q we take the double nodes technique in order to allow possible discontinuity of q and r at the edges.

The boundary $\Gamma = \partial\Omega$ is uniformly divided into (a) 48 and (b) 96 boundary elements respectively as shown in Fig. 2, where the nodes on the boundary are indicated by small black dots.



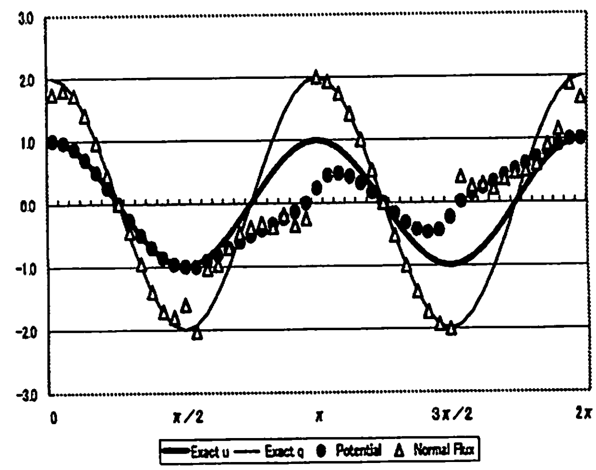
(a)



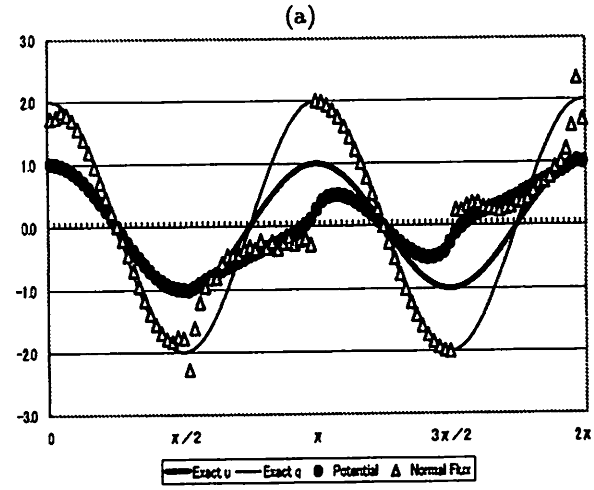
(b)

Fig. 2 Boundary elements

Calculated profiles of u^h and q^h against the central angle ϑ ($0 \le \vartheta < 2\pi$) are depicted in Fig.3 with reference to the exact u and q along the boundary Γ. The calculated q^h is in good agreement with the exact q on Γ_u, and the calculated u^h is in fairly good agreement with the exact u on Γ_q. The calculated u^h and q^h are deteriorated on $\Gamma \setminus (\Gamma_u \cup \Gamma_q)$.



(a)



(b)

Fig. 3 Exact and calculated u^h, q^h on the boundary

6 Conclusions

The under-determined problem is considered for the Laplace equation in two dimensions. By introducing a functional to be minimized, the solution of the inverse problem is understood as the minimizer of the functional. The necessary condition for the functional to attain the minimum is paraphrased by the primary and adjoint boundary value problems of the Laplace equation. The direct boundary element method is applied to obtain numerical solution of the problems, yielding an augumented system of linear algebraic equations. The linear system of equations can be solved directly. A test example merely suggested that the calculated results are acceptable in the vicinity of the boundary components where boundary data are prescribed.

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